ON ONE FORM OF EQUATIONS OF RIGID BODY ROTATIONAL MOTION IN S-PARAMETERS

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ABSTRACT

The investigation of attitude motion of rigid body under the action of principle moment \( \vec{M} \) of external forces is usually realized in terms of Euler angles, various variants of “aircraft” angles or direction cosines. The wide range of mechanical problems is connected with the studying of rotational motion of rigid body that is such attitude motion of the body with respect to the center of mass, when the work of moment \( \vec{M} \) is small with respect to the kinetic energy of the body rotational motion. These problems are usually investigated with the use of special variables. For example, Andoyer dynamical equations written in Beletsky-Chernousko variables are used widely in analytical investigations [1]. But the computer simulation of mathematical models constructed with the use of these variables meets some difficulties due to the presence of singularities. The use of Rodriguez-Hamilton parameters \( \lambda_i \) (\( i = 0 \ldots 3 \)) allow to avoid the mentioned difficulties. So these parameters are very popular in rigid body attitude dynamics [2], [3]. But they are not minimal in number. The new parameters \( s_i \) (modified Rodriguez-Hamilton parameters) are introduced in the paper [4] for construction the mathematical model of the body’s attitude motion. They possess the same advantages as the Rodriguez-Hamilton parameters and at the same time they are minimal in number. S-parameters may be considered as the result of stereographic projection of 4-dimensional sphere of quaternions, normalized with the condition \( \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \), on 3-dimensional hyper plane, orthogonal to the axis \( O\lambda_0 \). At the same time parameters \( s_i \) may be expressed in terms of Rodriguez-Hamilton parameters according to the formulae \( s_i = \lambda_i / (1 - \lambda_0) \). Choosing s-parameters and dimensionless projections of angular momentum on principal central axes of inertia as variables we obtain in the normal form the system of six differential equations (3 dynamical and 3 cinematic equations) for finding six unknowns [3]. The differential equations of the body’s attitude motion constructed in [3] are suitable both for analytical and numerical investigations of the rotary motion [5]. In this paper the new equations of the body’s attitude motion are deduced. They differ from analogous equations obtained in [3] because we use the angular velocity (the value and direction of angular velocity vector) instead of angular momentum. In some cases...
this approach is convenient for computer modeling of rigid body motion and more obvious in interpretation of research results.

KEY WORDS: COMPUTER MODELING OF RIGID BODY MOTION

1. The statement of the problem

The studying of attitude motion of rigid body under the action of principle moment $\vec{M}$ of external forces is usually realized in terms of Euler angles, various variants of “aircraft” angles or direction cosines. The wide range of mechanical problems is connected with the studying of rotational motion of rigid body that is such attitude motion of the body with respect to the center of mass $C$, when the work of moment $\vec{M}$ of external disturbing forces is small with respect to the kinetic energy of the body’s rotational motion. These problems are usually investigated with the use of special variables. For example, the Anduaye dynamical equations of rigid body attitude motion are used widely in Beletsky-Chernousko variables $L, \rho, \sigma, \varphi, \psi, \vartheta$ [1]:

$$
\frac{d\varphi}{dt} = L \cos \vartheta \left( \frac{1}{C} - \frac{\sin^2 \varphi}{A} - \frac{\cos^2 \varphi}{B} \right) + \frac{M_1 \cos \psi + M_2 \sin \psi}{L \sin \vartheta},
$$

$$
\frac{d\psi}{dt} = L \left( \frac{\sin^2 \varphi}{A} + \frac{\cos^2 \varphi}{B} \right) - \frac{M_1 \cos \psi + M_2 \sin \psi}{L} \cot \vartheta - \frac{M_2}{L} \cot \rho, 
$$

$$
\frac{d\vartheta}{dt} = L \left( \frac{1}{A} - \frac{1}{B} \right) \sin \vartheta \sin \varphi \cos \varphi + \frac{M_2 \cos \psi - M_1 \sin \psi}{L},
$$

$$
\frac{dL}{dt} = M_3, \quad \frac{d\rho}{dt} = \frac{M_1}{L}, \quad \frac{d\sigma}{dt} = \frac{M_2}{L \sin \rho}.
$$

Here $A, B, C$ are the body’s principal central moments of inertia, $L$ is the absolute value of kinetic moment $\vec{L}$, the angles $\rho, \sigma$ determine the position of vector $\vec{L}$ and connected with him orthogonal Cartesian coordinate system $CL_1L_2L_3$ with respect to the Koenig coordinate system $CXYZ$ according to Fig. 1, the Euler angles $\varphi, \psi, \vartheta$ determine the orientation of the body’s principal central axes of inertia $x, y, z$ with respect to the coordinate system $CL_1L_2L_3$, and $M_1, M_2, M_3$ are the projections of the moment $\vec{M}$ on the axes of moving coordinate system $CL_1L_2L_3$. 

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Fig. 1

The usability of these equations is connected with the constancy of vector $\mathbf{L}$ (that is the values $L, \rho, \sigma$) in undisturbed motion of the body. But the computer simulation of mathematical models constructed with the use of Euler angles meets the following difficulties:

1. the cinematic equations degenerates at certain values of angles.
2. the right parts of differential equations occurs complex trigonometric functions of angles, so the integration of these equations leads to significant increasing of computing time.

The Euler-Poisson model based on the use of direction cosines, is free of these disadvantages, but the closed system of equations in these variables have much more high order, so it is not convenient for computer modeling.

The mentioned above difficulties may be avoided by introducing new parameters. Among all cinematic parameters the significant place belongs to the Rodriguez-Hamilton parameters $\lambda_i$ ($i = 0…3$) [2]. They represent the components of 4-dimensional number $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, called quaternion, and connected by the normalization condition $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. Choosing these parameters and dimensionless projections of angular velocity on principal central axes of inertia as variables we obtain in the normal form the system of 7 differential equations (3 dynamical and 4 cinematic equations) for finding seven unknowns.

But the Rodriguez-Hamilton parameters are not minimal in number. The new parameters $s_1, s_2, s_3$ (s-parameters) are introduced in the paper [3] for construction the mathematical model of the body’s attitude motion. They possess the same advantages
as the Rodriguez-Hamilton parameters and at the same time they are minimal in number. S-parameters may be considered as the result of stereographic projection of 4-dimensional sphere of quaternions, normalized with the condition \( \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \), on 3-dimensional hyper plane, orthogonal to the axis \( O\lambda_0 \) (Fig 2).

At the same time parameters \( s_i \) may be expressed in terms of Rodriguez-Hamilton parameters according to the formulae \( s_i = \frac{\lambda_i}{1-\lambda_0} \). The differential equations of the body’s attitude motion suitable for the investigation of the rotary motion, were constructed in the paper [4] with the use of s-parameters. In this paper the new equations of the body’s attitude motion are deduced. They differs from analogous equations obtained in [4] with the use of variables \( L, \rho, \sigma, s_1, s_2, s_3 \) because instead of variables \( L, \rho, \sigma \) we use the variables \( \omega, \alpha, \beta \) which determine the value and direction of the body’s angular velocity vector and more obvious in interpretation of the results of investigation.

2. The equations of motion

Let us determine the orientation of the body with the use of moving coordinates \( C\omega_1\omega_2\omega_3 \) (Fig. 3), where the axis \( C\omega_1 \) is directed along the vector \( \vec{\omega} \) of body’s angular velocity vector and \( CXYZ \) is the Koenig coordinate system.
Let us introduce the direction cosines matrix $\mathbf{K}$ of the axes $\omega_1$, $\omega_2$, $\omega_3$ with respect to the axes $X$, $Y$, $Z$:

\[
\begin{align*}
\omega_1 & \quad \omega_2 & \quad \omega_3 \\
X & \quad \sin \alpha \cos \beta & \quad \cos \alpha & \quad \sin \alpha \sin \beta \\
Y & \quad -\sin \beta & \quad 0 & \quad \cos \beta \\
Z & \quad \cos \alpha \cos \beta & \quad -\sin \alpha & \quad \cos \alpha \sin \beta
\end{align*}
\]

and the direction cosines matrix $\mathbf{A}$ of the body’s principle central axes of inertia $x$, $y$, $z$ with respect to the axes $\omega_1$, $\omega_2$, $\omega_3$. The elements of matrix $\mathbf{A}$ can be expressed in terms of s-parameters $s_1$, $s_2$, $s_3$ [3]:

\[
\begin{align*}
\omega_1 & \quad \frac{(u_0 - 8(s_1^2 + s_2^2))}{u_0} & \quad \frac{4(2s_1s_2 - s_3(s^2 - 1))}{u_0} & \quad \frac{4(2s_1s_3 + s_1(s^2 - 1))}{u_0} \\
\omega_2 & \quad \frac{4(2s_1s_2 + s_3(s^2 - 1))}{u_0} & \quad \frac{(u_0 - 8(s_1^2 + s_2^2))}{u_0} & \quad \frac{4(2s_2s_3 - s_1(s^2 - 1))}{u_0} \\
\omega_3 & \quad \frac{4(2s_1s_3 - s_2(s^2 - 1))}{u_0} & \quad \frac{4(2s_2s_1 + s_1(s^2 - 1))}{u_0} & \quad \frac{(u_0 - 8(s_1^2 + s_2^2))}{u_0}
\end{align*}
\]

Where

\[
|s|^2 = s_1^2 + s_2^2 + s_3^2, \quad u_0 = (|s|^2 + 1)^2.
\]

The following equations are valid:

\[
(0, 0, \omega)^T = \mathbf{A}(\omega_1, \omega_2, \omega_3)^T
\]

\[
(M_X, M_Y, M_Z)^T = \mathbf{K}(M_{\omega_1}, M_{\omega_2}, M_{\omega_3})^T
\]
The absolute angular velocity vector we represent in the form:

$$\vec{\omega} = \vec{\omega}_c + \vec{\omega}_r$$

where $\vec{\omega}_c = \vec{\alpha} + \vec{\beta}$ is the angular velocity of the coordinate system $C\omega_1\omega_2\omega_3$, and $\vec{\omega}_r$ is the angular velocity of the body relative to the coordinate system $C\omega_1\omega_2\omega_3$.

Since

$$(\omega_{eω1}, \omega_{eω2}, \omega_{eω3})^T = (-\alpha \sin \beta, \beta, \alpha \cos \beta)$$

then on the basis of (2) and (3) we obtain:

$$(\omega_x, \omega_y, \omega_z)^T = \Lambda^T \left(\alpha \sin \beta, -\beta, \omega - \alpha \cos \beta \right)$$

The angular momentum vector $\vec{L}$ of the rigid body with tensor of inertia diag$(A, B, C)$ (in the axes $Cxyz$) has the following projections:

$$(L_{eω1}, L_{eω2}, L_{eω3})^T = \Lambda \cdot \text{diag}(A, B, C) \cdot (\omega_x, \omega_y, \omega_z)^T = \Lambda \cdot (0, 0, \omega)^T$$

$$\Lambda = \Lambda \cdot \text{diag}(A, B, C) \cdot A^T,$$  \hspace{1cm} \Lambda_i = \Lambda \cdot (0, 0, 1)^T$$

Differentiating expression (4) with respect to time, we can write the theorem of angular momentum $\frac{d}{dt} (L_x, L_y, L_z)^T = (M_x, M_y, M_z)^T$ in the form:

$$\dot{\omega} \Lambda + \omega \dot{\Lambda} + \omega \dot{\Lambda} = (M_{eω1}, M_{eω2}, M_{eω3})^T$$

By left multiplying the equation (5) by $K^T$, we obtain:

$$\left(\dot{\omega}E + \omega K^T \dot{K}\right)\Lambda + \omega \dot{\Lambda} = (M_{eω1}, M_{eω2}, M_{eω3})^T$$

Let us introduce into consideration matrices

$$\Omega_ω = \begin{pmatrix} 0 & -\omega_{eω3} & \omega_{eω2} \\ \omega_{eω3} & 0 & -\omega_{eω1} \\ -\omega_{eω2} & \omega_{eω1} & 0 \end{pmatrix} \quad \text{and} \quad \Omega_x = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

Then

$$\dot{K} = KΩ_ω, \quad \dot{A} = AΩ_x$$
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Consequently,
\[
\dot{\mathbf{A}} = \frac{d}{dt}(\mathbf{A}\text{diag}(A, B, C)\mathbf{A}^T) = \mathbf{A}\text{diag}(A, B, C)\mathbf{A}^T + \mathbf{A}\text{diag}(A, B, C)\dot{\mathbf{A}}^T = \\
= \mathbf{A}\Omega_{\times}\text{diag}(A, B, C)\mathbf{A}^T + \mathbf{A}\text{diag}(A, B, C)(-\Omega_{\times})\mathbf{A}^T =
\]
\[
= \mathbf{A}\left[\Omega_{\times}\text{diag}(A, B, C)\mathbf{A}^T - \text{diag}(A, B, C)\Omega_{\times}\right]\mathbf{A}^T = \mathbf{A}\mathbf{A}_{\Omega}\mathbf{A}^T
\]
where
\[
\mathbf{A}_{\Omega} = \begin{pmatrix}
0 & (A-B)\omega_{rz} & (C-A)\omega_{ry} \\
(A-B)\omega_{rz} & 0 & (B-C)\omega_{rx} \\
(C-A)\omega_{ry} & (B-C)\omega_{rx} & 0
\end{pmatrix}
\]
Therefore the equation (6) may be written in the form:
\[(7) \quad (\omega \mathbf{A}_{\omega} + \dot{\omega}\mathbf{E})\mathbf{A}_{\Omega} + \omega\mathbf{A}\mathbf{A}_{\Omega}\mathbf{A}^T(0, 0, 1)^T = (M_{\omega 1}, M_{\omega 2}, M_{\omega 3})^T
\]
The system (7) is linear with respect to \(\dot{\alpha}, \dot{\beta}, \dot{\omega}\) and may be resolved for these derivatives. However, before this operation we shall reduce the equations (7) to the form, which containing \(M_{\times}, M_{y}, M_{z}\) instead of \(M_{\omega 1}, M_{\omega 2}, M_{\omega 3}\), and is more convenient for the solving of the problems where the perturbing moment \(\vec{M}\) is defined by the projections on the axes, connected with the rigid body. For this purpose we can multiply (7) from the left by \(\mathbf{A}^T\):
\[
\left[\mathbf{A}^T(\omega \mathbf{A}_{\omega} + \dot{\omega}\mathbf{E})\mathbf{A} + \omega\mathbf{A}\mathbf{A}_{\Omega}\mathbf{A}^T\right](0, 0, 1)^T = (M_{\times}, M_{y}, M_{z})^T
\]
Thus, we obtained the linear system with respect to \(\dot{\alpha}, \dot{\beta}, \dot{\omega}\). By resolving it for the derivatives we obtain:
\[(8) \quad \begin{pmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\omega}
\end{pmatrix} = \mathbf{A}^T\begin{pmatrix}
\frac{M_{\times}}{A} \\
\frac{M_{y}}{B} \\
\frac{M_{z}}{C}
\end{pmatrix} + \omega\mathbf{B} \left[(-B)(A-C)A_3 - (C-A)A_2 - (A-B)A_1 , 0 \right],
\]
where \(A_{ij}\) are the elements of the direction cosines matrix \(\mathbf{A}\) (1).

For closing the equations (8) we consider them together with cinematic equations, which have the following form in s-parameters:
\[
(\dot{s}_1, \dot{s}_2, \dot{s}_3)^T = \mathbf{B}(\omega_{rx}, \omega_{ry}, \omega_{rz})^T,
\]
Here \( B = -\frac{1}{4} \begin{pmatrix} s_1^2 - s_2^2 - s_3^2 + 1 & 2(s_1s_2 + s_3) & 2(s_1s_3 - s_2) \\ 2(s_1s_2 - s_3) & -s_1^2 + s_2^2 - s_3^2 + 1 & 2(s_2s_3 + s_1) \\ 2(s_1s_3 + s_2) & 2(s_2s_3 - s_1) & -s_1^2 - s_2^2 + s_3^2 + 1 \end{pmatrix} \)

On the basis of (3) and the easily verified equation \( BA^T = \frac{u_0}{16} \) we obtain the cinematic equations in \( s \)-parameters:

\[
\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \\ \dot{s}_3 \end{bmatrix} = B^T \begin{bmatrix} \alpha \sin \beta \\ -\dot{\beta} \\ \omega - \dot{\alpha} \cos \beta \end{bmatrix}
\]

Thus, the system (8), (9) represent the differential system for description the rotational motion of rigid body. This system is convenient for computer modeling because it has the unique singular point – the pole of function, mapping the unit sphere in 4-dimensional quaternion space onto the 3-dimensional hyperplane of \( s \)-parameters. But the direct hit of image point into the pole of mapping function during the motion is the exception case, demanding special initial conditions. The set of such initial conditions in the space of \( s \)-parameters has the zero measure and so the possibility of hitting the image point into the pole of mapping function practically does not realize.

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