THERMAL INSTABILITY OF THIN VISCOUS FILMS

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Abstract. In the present work the dynamics and heat transfer of a thin viscous film, with fully mobile interfaces is studied in the case when the inertial, viscous, capillary, van der Waals and thermocapillary forces are important. The film is laterally bounded by a frame, whose temperature is higher than that of the environment around the film. The stability of the static film shapes is examined by a linear and non-linear analysis. The results show that the film rupture is mostly governed by the dynamics, but it could be delayed or enhanced by the thermocapillary convection and the heat transfer to the surrounding environment.

Key words: free thin film, heat transfer, thermocapillary convection, van-der-Waals force, linear stability, non-linear stability

*One of the authors (S.R.) kindly acknowledges the financial support of Bulgarian Research Fund in the frames of the Project DCVP 02/1.
1. Introduction

The isothermal behavior of thin films (free or supported) has been extensively investigated for many years by different authors [1], [2]. However, the non-isothermal case has received less attention although the variety of its applications in the modern technologies: ink-jet printing (the thin films could be regarded as a Cartesian approach of the thin jets); large glass sheets for the LCD displays; solidifying slices of semiconductor materials for the electronics microchips; etc.

The influence of the free film dynamics on the heat transfer and solidification at a radiation transfer from the film interfaces has been studied numerically in [3], [4]. In a recent paper [5] the long-wave analysis has been applied for an infinite free viscous sheet with a given temperature different from the ambient one. The authors show that there exists a stable thermal mode coupled to the dynamical modes obtained by linear stability analysis. This coupling can promote or delay the onset of the film rupture, if a convenient phase shift to the initial temperature profile with respect to the initial velocity profile is applied.

In the present work, the non-isothermal free film dynamics is studied. A linear stability analysis is implemented numerically by the method of the differential Gauss elimination like in some previous our works [6], [7]. It is found that the film will be less stable with the increase of the thermocapillary convection and heat transfer with the surrounding gas. Similar results are obtained by the non-linear stability analysis, from which also the evolution of the film thickness, lateral velocity and temperature are found.

2. Problem statement

The film fluid is supposed Newtonian viscous with constant physical properties: density $\rho$, dynamic viscosity $\mu$, thermal conductivity $\kappa$, heat capacity $c$ and heat transfer coefficient with the ambient gas $\beta$. The surface tension $\sigma$ depends on the film temperature $\theta$ as: $\sigma = \sigma_0 - \gamma(\theta - \theta_0)$, where $\gamma$ is the rate of surface tension dependence on temperature. The film drains because of inertia, viscosity, thermocapillary convection and van-der-Waals force acting as a disjoining pressure. The heat transfer is supposed to be due to conduction, convection inside the film and heat transfer with the surrounding gas, whose temperature is $\theta_0$. The film is assumed symmetrical with respect to its central plane $z = 0$ and laterally bounded by a solid rectangular frame, hotter than the ambient. Thus the frame temperature $\theta_1$ is different from $\theta_0$, i.e. $\theta_1 > \theta_0$. The film can be considered as one dimensional (in $x$ direction), if one
side of the frame is much longer than the other. The average film thickness $\bar{h}$ is assumed to be much smaller than the shorter frame side $a$, such that $\bar{h} = \varepsilon a$, where $\varepsilon \ll 1$. The free symmetrical surfaces are defined as $z = \pm h'/2$, where $h'(x,t) = O(\varepsilon)$ represents the film shape. Next, the following characteristic scales will be used: $a$ for the lateral length, $U$ for the lateral velocity, $a/U$ for the time, $\Delta \theta = (\theta_1 - \theta_0)$ for the temperature and $\varepsilon a$ for the film thickness.

The coupled thermal-dynamic system in dimensionless form for the film thickness $h = h'/\varepsilon a$, longitudinal velocity $u$ and temperature $T = (\theta - \theta_0)/(\theta_1 - \theta_0)$ is given by (similarly as in [3], [4], [5]):

\begin{equation}
\tag{2.2}
\frac{\partial h}{\partial \tau} + \frac{\partial}{\partial x}(uh) = 0,
\end{equation}

\begin{equation}
\tag{2.3}
\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} = \frac{\varepsilon}{We} \frac{\partial^3 h}{\partial x^3} + \frac{4}{Re} h \frac{\partial}{\partial x} \left( h \frac{\partial u}{\partial x} \right) + \frac{A}{h^4} \frac{\partial h}{\partial x} - \frac{2M}{h} \frac{\partial T}{\partial x},
\end{equation}

\begin{equation}
\tag{2.4}
\frac{\partial T}{\partial \tau} + u \frac{\partial T}{\partial x} = 1 \frac{\partial}{Peh \partial x} \left( h \frac{\partial T}{\partial x} \right) - \frac{Bi}{Peh} T,
\end{equation}

where $\tau$ is the dimensionless time, $Re = \rho a U/\mu$ is the Reynolds number, $We = 2\rho a U^2/\sigma_0$ is the Weber number, $A = A_H/2\pi \rho U^2 a^3 \varepsilon^3$ is the dimensionless Hamaker constant ($A_H$ being the dimensional Hamaker constant), $M = \frac{1}{\varepsilon We} \frac{\gamma}{\sigma_0} \Delta \theta$ is the Marangoni number, $Pe = Pr Re$ is the Peclet number, $Pr = c \mu / \kappa$ is the Prandtl number and $Bi = \frac{2\beta a}{\varepsilon \kappa}$ is the Biot number.

The boundary conditions for $h$, $u$ and $T$ on the frame are the wetting condition for the film shape $h$, the no-slip condition for the lateral velocity $u$ and the constant temperature condition, respectively given by:

\begin{equation}
\tag{2.5}
\frac{\partial h}{\partial x}(\pm 1, \tau) = \pm \tan \alpha, \quad u(\pm 1, \tau) = 0, \quad T(\pm 1, \tau) = 1,
\end{equation}

where $\pi/2 - \arctan(0.5 \tan \alpha)$ is the wetting angle with the frame (Note, that for planar films $\alpha = 0$, as the wetting angles are right angles.).

The mass (volume) conservation condition reads:

\begin{equation}
\tag{2.6}
\int_{-1}^{1} h \, dx = W,
\end{equation}
where \( W = \text{const} \) is the initial film volume (here taken \( W = 2 \)).

Since the system (2.2)–(2.4) is of \( O(\varepsilon) \), the dimensionless numbers are limited as: \( Re \leq \varepsilon^{-1}, We \leq 1 \) and \( Pe \leq \varepsilon^{-1} \). Usually the water films of mean thickness of \( O(10^{-5} m) \) attached on frames with sides of \( O(10^{-2} m) \) enter into these restrictions.

3. Results

3.1. Film relaxation and drainage

If the complementary wetting angle is zero, \( \alpha = 0 \), then there exists an analytical static solution of the system (2.2)–(2.4): \( h_s = 1, T_s = 1 \) in the adiabatic case, i.e. at \( Bi = 0 \). However, for \( \alpha > 0 \), the static solution can only be obtained as a numerical solution of the dynamical problem with some initial conditions, for example: \( h_0(x) = 1, u_0(x) = 0, T_0(x) = 0 \). It is found that the static shapes exist only for some combinations of the problem parameters (for some values of \( B, \alpha, Ma \) and \( Bi \), where \( B = WeA/\varepsilon \) and \( Ma = WeM/\varepsilon \) [6]). For example, at \( B \geq 9 \), the static film shapes exist only for \( \alpha = 0 \), while at \( B = 0.01 \) – for \( 0 \leq \alpha \leq 1.37 \). If the film shape is static for \( Ma = Bi = 0 \), it remains also static at different values of \( Ma \) and \( Bi \). Contrary, if at \( Ma = Bi = 0 \) the film ruptures, the increase of \( Ma \) and \( Bi \) increases the rupture time and even a static shape could be reached. An example is shown in Fig. 1 for \( \alpha = 0.85 \) at \( Ma = 0, Bi = 0 \) (plot (a)) and \( Ma = 1, Bi = 1 \) (plot (b)). The rupture time is \( \tau = 48.4 \) for the case on plot (a) and the relaxation time (the time of the static shape reach) is \( \tau = 6.8 \) for the case on plot (b). We have to note that the rupture time at \( M = 0 \) remains the same with the change of \( Bi \), as the heat transfer is separated from the film dynamics. The increase of \( M \) leads to a competition between the Marangoni stresses and van-der-Waals forces, as the latter are responsible for the film rupture.

3.2. Linear stability

Small disturbances \( \tilde{h}(x, \tau), \tilde{u}(x, \tau) \) and \( \tilde{T}(x, \tau) \) are imposed on the static state solutions:

\[
(3.9) \quad h = h_s + \tilde{h}, \quad u = \tilde{u}, \quad T = T_s + \tilde{T}.
\]

The disturbances \( \tilde{h}, \tilde{u} \) and \( \tilde{T} \) are sought as exponential functions in time with unknown growth rate \( \omega = \omega_r + i\omega_i \) and unknown distributions \( H(x), V(x) \) and \( G(x) \):

\[
(3.10) \quad \tilde{h} = H(x)e^{\omega \tau}, \quad \tilde{u} = V(x)e^{\omega \tau}, \quad \tilde{T} = G(x)e^{\omega \tau}.
\]
After inserting (3.9) with (3.10) in Eqs (2.2)-(2.4), neglecting the 2–nd order disturbances, a characteristic system for \( \omega \) is obtained. At \( \alpha = 0 \) and \( Bi = 0 \), it turns into an algebraic equation of 3\(^{rd}\) degree for \( \omega \), independent of the Marangoni number \( M \), with one negative root connected to the thermal mode, \( \omega_1 = -m^2/Pe \) and two other roots (a negative one and the other positive or negative, depending on \( B \)), \( \omega_{2,3} = \frac{-2m^2}{Re} \left[ 1 \mp \sqrt{1 - \frac{\varepsilon Re^2(m^2 - B)}{4We m^2}} \right] \), where \( m \) is the wave number. The cutoff wave number \( m_c = \sqrt{B} \), at which \( \omega_2 = 0 \), is the same as it has been found in the isothermal case by many authors [8] and also in our previous works [6], [7]. Thus, for the plane static films, the thermal instability always decays in time, while the film shape instability may rupture the film.

In the case of \( \alpha > 0 \) the characteristic system for \( \omega \) has no analytical solution. It is solved numerically by the secant method and the differential Gauss elimination method that have been applied to the case of an isothermal film in [6], [7]. For example, at \( Re = Pe = 1 \), \( We = \varepsilon = 0.01 \) and \( B = 3 \), the numerically found eigenvalues \( \Omega_r = \frac{\omega_r We}{\varepsilon Re} \) (the imaginary part is zero, \( \omega_i = 0 \)) are presented in Fig. 2 as functions of \( \alpha \), \( Ma \) and \( Bi \). The value of \( \Omega_r = 0.1314 \) at \( \alpha = 0 \), \( Ma = 0 \) and \( Bi = 0 \) is confirmed by the root \( \omega_2 \) at \( m = \pi/2 \) (\( m = \pi/2 < m_c = \sqrt{3} \) corresponds to the unstable mode). From these results, it is clear that for all values of \( M = 0; 0.1; 1 \) and \( Bi = 0; 0.1; 1 \) the film is unstable, except for \( \alpha = 0.6 \) at \( Bi = 0 \) or at \( Bi > 0 \) and \( Ma < 1 \).
bigger $Ma$, i.e., the higher temperature difference $\Delta \theta$, will promote instability, i.e. rupture process.

3.3. Non-linear stability

The non-linear stability analysis is based on the numerical solution of the thermo-dynamical problem (2.2)-(2.6) with different initial conditions. Here, we shall concentrate on those initial conditions correspondent to the "most dangerous" wave number $m = \pi/2$, for example the asymmetrical disturbances applied only on the film shape:

$$(3.11) \quad h_0(x) = h_s(x) + 0.1 \sin(\pi/2x), \quad u_0(x) = 0, \quad T_0(x) = T_s(x).$$

For the case shown in Fig. 2 the predictions of the linear stability analysis are approximately confirmed by the non-linear analysis results. In Fig. 3a), b) the film thickness rupture is shown at $Re = Pe = 1$, $We = \varepsilon = 0.01$, $A = 3$ ($B = 3$), $Ma = 1$ and $Bi = 1$ for $\alpha = 0.6$ and $\alpha = 0.8$, respectively. The rupture takes place at different points $x$ and times $\tau$: at $x = -0.525$ and $\tau = 25.2$ for $\alpha = 0.6$ and at $x = -0.315$ and $\tau = 37.9$ for $\alpha = 0.8$. We
performed also calculations with other initial disturbances and found a delay of the rupture or a rapid return to the static shape. For example, at the upper given values of the parameters \(Re = Pe = 1, We = 0.01, \varepsilon = 0.01, A = 3 \ (B = 3), M = Ma = 1, Bi = 1\) after applying the initial disturbance \(h(x, 0) = h_s(x) + 0.1 \sin(\pi/2x)\) on its static shape \(h_s(x)\) and at: a) \(\alpha = 0.6\); b) \(\alpha = 0.8\). (The final states are plotted as "dash-dotted" lines, the initially disturbed states - as "dotted" lines.)

4. Conclusions

In the present paper we study the dynamics of a non-isothermal free thin viscous film, attached on a rectangular frame, and the role of the heat transfer on its stability. The van-der-Waals attractive disjoining pressure and Marangoni thermo-capillary convection are taken into account. An evolutionary system consisting of three coupled non-linear PDE for the film thickness,
lateral velocity and temperature describes the considered thermo-dynamical process. In the general case of the problem parameters, its solution has been obtained numerically. Since the film relaxation leads to two different final states: static film shape and film rupture, the increase of the Marangoni number could change the rupture state into a static one when the other parameters are left the same. For the obtained static shapes, a linear stability analysis is implemented numerically by the method of the differential Gauss elimination. It is found that the instability growth rate is positive and increases with the increase of the Marangoni number for almost all the wetting angles at the considered values of the parameters. These results are approximately confirmed by the results of the non-linear stability analysis, from which the evolution of the film thickness, lateral velocity and temperature distribution are found. The importance of the wetting angle is confirmed: the film can be stable or unstable depending on the competition between the wetting with the lateral boundary (frame), the thermo-capillary convection and the van-der-Waals attraction. An idea to control the rupture process by appropriate initial thermal disturbances is proposed, which will be further developed in our future studies.

REFERENCES