MODELING OF CYLINDRICAL COUETTE FLOW OF RAREFIED GAS. THE CASE OF ROTATING INNER CYLINDER

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ABSTRACT. The cylindrical Couette flow of a rarefied gas is studied in the case when the inner cylinder is rotating while the outer cylinder is at rest. Velocity, density and temperature profiles are investigated by a Direct Monte Carlo Simulation method and a numerical solution of the Navier-Stokes equations for compressible flow is found. The results obtained by both methods are: in an excellent agreement at a small Knudsen number Kn=0.02; in a satisfactory agreement at Kn=0.1 and they vastly differ each other at a moderate Kn=0.5. The comparison shows that the continuum approach can be used successfully for calculations of nonisothermal rarefied gas flows at small Knudsen numbers Kn<0.1. These results are important for applications in non-planar microfluidic problems.

KEY WORDS: Kinetic theory, Rarefied gas, Microfluidics, DSMC

1. Introduction

The Couette cylindrical flow is a fundamental problem in the rarefied gas dynamics [1,9,10,11]. As such, its modeling and numerical solving is of a great importance for the microfluidics, which is the theoretical background for analysis of new emerging Micro Electro Mechanical Systems MEMS [2,3,12,14].

The design of adequate mathematical models of gaseous flows in micro devices is one of the most important tasks of the studies. We consider both molecular
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and continuum models treating the gaseous flow by using different level of mathematical description. Both models take into account the specific microfluidic effects of gas rarefaction and slip-velocity regime at the solid boundaries [6,7,8].

In the present paper we compare results obtained by using the molecular Direct Simulation Monte Carlo (DSMC) method with those calculated by a numerical solution of the continuum Navier-Stokes equations for compressible flow (NS). The aim of the paper is delineate the range of validity of the NS with respect to the DSMC solution, which convergences to the Boltzmann equation when the number of modeling particles is large enough to be considered as tended towards infinity.

2. Formulation of the problem and methods of solution

We study a rarefied gas between two coaxial cylinders (one dimensional, axis-symmetrical problem) with equal temperatures $T_1=T_2$. The inner cylinder has radius $R_1$ and the outer – $R_2$. The inner cylinder rotates with a constant velocity $V_1$ and the outer one is static – Figure 1.

2.1 Continuous Model and Numerical Simulation

The continuous model is based on the Navier-Stokes equations for compressible Newtonian fluid, completed with the equations of continuity and energy transport. The governing equations are written in dimensionless form as follows:

\[
\begin{align*}
(2.1) & \quad \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho u) = 0 \\
(2.2) & \quad \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} - \frac{v^2}{r} \right) = -\frac{\partial P}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} (r \tau_n) - \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{ro}}{r} \\
(2.3) & \quad \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + uv \right) = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{ve}) - \frac{\partial \tau_{ze}}{\partial z} + \rho g_v \\
(2.4) & \quad \rho c_p \frac{DT}{Dt} = \text{div}(\lambda \text{grad} T) - p \text{div} \mathbf{V} + \mathbf{\rho \Phi}
\end{align*}
\]
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\( P = \rho RT \)

Where \( \mathbf{V} \) is the velocity vector, \( u \) and \( v \) are the velocity components along axis \( r \) and \( \phi \). A rather standard notation is used in Eqs. (2.1)-(2.5): \( \rho \) is density and \( T \) is the temperature. \( \rho, P, T, u, v = f(r, t) \). The therms \( \tau_{i,j} \) are the stress tensor components and \( \Phi \) is the dissipation function [15]. The equations are normalized by using the following scales: for density, \( \rho_0 = mn_0 \), for velocity, \( V_0 = \sqrt{2RT_0} \), \( R \) is the gas constant, for length - the distance between the cylinders \( L = R_2 - R_1 \), for time \( t_0 = L/V_0 \), for temperature \( T_0 = T_w \) - the wall temperature of both cylinders.

For a perfect monatomic gas, the viscosity and the coefficient heat transfer read as [13]:

\[
(2.6) \quad \mu = \mu(T) = C_\mu \rho_0 V_0 \sqrt{T}, \quad C_\mu = \frac{5}{16} \sqrt{2\pi}
\]

\[
(2.7) \quad \lambda = \lambda(T) = C_\lambda \rho_0 V_0 \sqrt{T}, \quad C_\lambda = \frac{15}{32} \sqrt{2\pi}
\]

The Knudsen number in (2.2)-(2.5) is \( Kn = l_0/L \), where the mean free path is \( l_0 \) and \( \gamma = c_p/c_v = 5/3 \).

For the problem (2.1)-(2.5) and \( t > 0 \), first-order slip boundary conditions are imposed at both walls, which can be written as follows [13,14]:

\[
(2.8) \quad u \mp 1.1466Kn \left( \frac{\partial u}{\partial r} - \frac{u}{r} \right) = \bar{u}_i
\]

\[
(2.9) \quad v = 0
\]

\[
(2.10) \quad T \pm 2.1904Kn \frac{\partial T}{\partial r} = \bar{T}_i,
\]

at \( r=\bar{r}_i \), : : \( i=1, 2 \). In Eqs. (2.8)-(2.10) \( \bar{u}_i = u_{i,0}/V_0 \) and \( \bar{T}_i = T_{w}/T_0 = 1 \) are the dimensionless wall velocity and temperature for both cylinders.

The equations of transfer (2.1)-(2.4), together with the boundary conditions (2.8)-(2.10) and zero initial distributions for \( u, v \) and \( T \), formulate the initial non-steady boundary-value problem. A second order of approximation, implicit difference scheme to solve numerically the formulated problem is used. Starting from the inner cylinder wall M grid knots are introduced along the coordinate \( r \). Thus, the difference value problem, for a given time \( t \), is reduced to the solution of 4 linearized systems of M algebraic equations.

2.2 Direct Simulation Monte Carlo (DSMC) Method

The gas considered is simulated as a stochastic system of \( N \) particles [4,5]. All quantities used are non-dimensional, so that the mean free path at equilibrium is equal to 1. The basic steps of simulation are as follows:
Fig. 2. Density (A, B, C) and ϕ velocity (D, E, F) profile for inner cylinder rotating
1 – \( V_1 = 0.3 \); 2 – \( V_1 = 0.5 \); 3 – \( V_1 = 0.7 \); 4 – \( V_1 = 1.0 \)

A. The time interval \([0; t^\hat{\text{t}}]\) over which the solution is found, is subdivided
into subintervals with step \( \Delta t \).

B. The space domain is subdivided into cells with sides \( \Delta z, \Delta r \).

C. Gas molecules are simulated in gap G using a stochastic system of \( N \)
points (particles) having position \( z_i(t), r_i(t) \) and velocities \( \overline{\xi}_i(t) \)
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D. $N_m$ particles are located in the $m$-th cell at any given time. This number varies during the computer simulation by the following two stages:

Stage 1. Binary collisions in each cell are calculated, whereas particles do not move. Collision modeling is realized using Bird’s scheme “no time counter”.

Stage 2. Particles move with new initial velocities acquired after collisions, and no external forces act on particles. No collisions are accounted for at this stage.

E. Stage 1 and Stage 2 are repeated until $t = t^\ast$.

F. Flow macro-characteristics (density, velocity, temperature) are calculated as time-averaged when steady regime is attained.

G. Boundary conditions are diffusive over the cylinders and periodical along axis $Oy$.

The modeling particles number for DSMC method is 3200000.

Fig. 3. Temperature profile for inner cylinder rotating velocity

1 – $V_1$=0.3; 2 – $V_1$=0.5; 3 – $V_1$=0.7; 4 – $V_1$=1.0
3. Numerical results

We study four cases at each of the Knudsen number – 0.02, 0.1 and 0.5, where the radius of the outer cylinder \( R_2 = 2 \) is twice larger than that of the inner one \( R_1 = 1 \), and different rotation velocities of the inner cylinder are considered: \( V_i = 0.3; 0.5; 0.7; 1.0 \). The outer cylinder is stationary.

All tests are implemented on the four Grid clusters in the Institute of Mechanics and the Institute for Parallel Processing, Bulgarian Academy of Sciences, which totally have more than 300 CPUs. These clusters are included in the Pan-European (EGEE) grid infrastructure. (see: www.eu-egee.org).

The results obtained by both methods are: in an excellent agreement at a small Knudsen number Kn=0.02; in a satisfactory agreement at Kn=0.1 and they vastly differs each other at a moderate Kn=0.5. The comparison shows that the continuum approach can be used successfully for calculations of non-planar isothermal rarefied gas flows at small Knudsen numbers Kn<0.1. The using of simplified boundary conditions or boundary conditions, which content second derivative along \( r \), not leads to the results improving.

REFERENCES