STABILITY OF A THERMOCONVECTIVE FERROFLUID FLOW IN A HORIZONTAL CHANNEL WITH RESPECT TO TWO-DIMENSIONAL WAVES

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ABSTRACT. A thermoconvective ferrofluid flow exists in a horizontal channel subjected to a longitudinal temperature gradient and to a strong oblique magnetic field [1]. The linear stability of that flow with respect to two-dimensional normal waves is studied.

KEY WORDS: ferrofluid, magnetic field, thermoconvective flow, linear stability, Galerkin method.

1. Introduction

The stability of parallel convective viscous flows in a horizontal layer subjected to a lateral heat gradient was first studied by Gershuni and Zhukhovitskii (see book [2] and references therein) for usual Newtonian fluids. When the liquid layer is a ferrofluid, subjected to a magnetic field, an additional force, the Kelvin force, reflecting the magnetization of the ferrofluid [3],[4],[5] intervenes in the usual Rayleigh-Benard convection. Moreover, for a strong magnetic field, the magnetic potential depends also on the temperature gradient.

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Recently, Hennenberg et al. [1] studied thermoconvective flows of a ferrofluid in a laterally heated channel subjected to an oblique strong magnetic field. The solution of the hydrodynamic problem was found in term of a power series in a small parameter, that measures the relative deviation of the magnetization across the layer. Limiting ourselves to the first-order term of the series, the basic flow is a two-dimensional parallel flow. Its linear stability has been formulated in [6] where preliminary results have been obtained. Here, we extend the stability analysis for more precisely boundary conditions along the channel walls.

2. Formulation of the problem

Let us consider a ferrofluid between two horizontal solid plates apart by a distance \(2d\), laterally heated and subject to an oblique strong magnetic field. The temperature along both plates and the external magnetic field are given by:

\[
T_w = T_r - \beta x, \quad \mathbf{H}^e = H^e (\cos \phi \mathbf{1}_x + \sin \phi \mathbf{1}_z) = H^e \mathbf{1}_H,
\]

where \(T_w\) is the plate temperature, \(T_r\) is the reference temperature at \(x = 0\), \(\beta > 0\), \(\phi\) is the inclination angle and \((\mathbf{1}_x, \mathbf{1}_z)\) are horizontal and normal unit vectors. The axis \(x\) is directed along the middle plane between the plates and the axis \(z\) is directed upwards. If the magnetic field vector is directed upwards, \(0 \leq \phi \leq \pi\) and when it is directed downwards, \(\pi < \phi \leq 2\pi\).

For strong magnetic fields, the magnetization \(M\) becomes equal to its saturation value which is independant upon the magnetic field, so that near the reference state \(M_r = M(T_r)\) is [4],[5]:

\[
M = M_r - K(T - T_r), \quad K = -\left(\frac{\partial M}{\partial T}\right)_H.
\]

The pyromagnetic coefficient \(K\) can be approximated as \(K \approx \alpha M\) where \(\alpha\) is the thermal expansion coefficient. The Maxwell equations are [1]:

\[
\nabla \times \mathbf{H} = 0, \quad \nabla \cdot \mathbf{H} = K(\nabla T \cdot \mathbf{1}_H).
\]

They obey the usual boundary conditions: the normal component of \(\mathbf{H} + \mathbf{M}\) and the tangential component of the magnetic field \(\mathbf{H}\) are continuous across the channel walls:

\[
H_z + M_z = H^e \sin \phi, \quad H_x = H^e \cos \phi.
\]
In the Boussinesq approximation, the incompressible thermoconvective flow in the channel obeys the mass, momentum and energy balance equations:

\[ \nabla \cdot \mathbf{v} = 0, \]

\[ \rho_0 \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} - \rho_0 g [1 - \alpha (T - T_r)] \mathbf{1}_z + \mu_0 M \nabla H, \]

\[ \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T, \]

where \( \mathbf{v} \) is the velocity, \( p \) the generalized pressure \[4\], \( \eta \) the dynamic viscosity, \( \rho_0 \) the density at \( T = T_r \), \( \kappa \) the thermal diffusivity of the liquid, \( \mu_0 \) the magnetic permeability in vacuum, and \( g \) the gravity acceleration. The momentum balance (6) includes also the Kelvin term \( \mu_0 M \nabla H \) expressing the magnetic force. The solutions of (5)-(7) satisfies the boundary conditions \( \mathbf{v} = 0 \) and \( T = T_w \) along the walls, and the condition of zero net flux across the channel, \( \int_0^d (\mathbf{v} \cdot \mathbf{1}_x) dz = 0 \).

We adimensionalize the problem scaling length, time, velocity, temperature and magnetic field by \( d, d^2/\kappa, \kappa/d, \beta d \) and \( H_e \), respectively. This introduces four parameters: the Prandtl number \( \text{Pr} \), the thermal Rayleigh number \( \text{Ra} \), the magnetic Rayleigh number \( \text{Ra}_m \) and a magnetization parameter \( \varepsilon_H \) whose definitions are:

\[ \text{Pr} = \frac{\nu}{\kappa}, \quad \text{Ra} = \frac{\rho_0 g \alpha \beta d^4}{\eta \kappa}, \quad \text{Ra}_m = \frac{\mu_0 K^2 \beta^2 d^4}{\eta \kappa}, \quad \varepsilon_H = \frac{K \beta d}{H_e} \ll 1. \]

The parameter \( \varepsilon_H \) characterizes the magnetization drop due to the temperature gradient, relative to the magnetic field. It is a small quantity, so that the solution is developed as a power series in it, limited to the first-order approximation. The stationary solution of (5)-(7) and (3) is \[1\]:

\[ \mathbf{v}_{st} = \frac{\kappa}{d} u_0(z) \mathbf{1}_x, \quad T_{st} - T_r = -\beta d [x + T_0(z)], \quad H_{st} = H^e h_0(z), \]

where the dimensionless functions \( u_0(z), h_0(z) \) and \( \theta_0(z) \) are given by

\[ \begin{align*}
  u_0(z) &= \mathcal{H}(\phi) U(z), \\
  T_0(z) &= \mathcal{H}(\phi) \theta(z), \\
  h_0(z) &= 1 - \varepsilon_H [z \sin 2\phi + T_0(z) \sin^2 \phi + \text{const}],
\end{align*} \]

\[ \mathcal{H}(\phi) = 1 + \frac{\text{Ra}_m}{\text{Ra}} \sin 2\phi, \]
The functions \( U(z) = -\frac{Ra}{6}z(z^2 - 1) \) and \( \theta(z) = -\frac{Ra}{360}(3z^5 - 10z^3 + 7z) \) appear in the Birikh problem for Newtonian viscous flow \([1, 2]\). They satisfy the following conditions: \( U(-1) = U(1) = 0 \), \( \int_{-1}^{1} U(z)\,dz = 0 \) and \( \theta(-1) = \theta(1) = 0 \). For \( \phi = 0 \) and \( \phi = \pi \), the magnetic field has no influence, and we recover the classical solution of the Newtonian fluid \([1, 2]\).

3. Stability analysis: Results and Discussion

We assume the validity of Squire’s theorem to study the linear stability of the basic flow with respect to two-dimensional disturbances. Thus, instead of the velocity \( v(u, w) \), we will consider the stream function \( \psi \), where \( u = \partial \psi / \partial z \) and \( w = -\partial \psi / \partial x \). Each perturbation will be of the form \( f = F(z) \exp(-\omega t + ikx) \), where \( F(z) \) is the amplitude of the stream function \( \Psi \), temperature \( \Theta \) and magnetic potential \( \Phi \). The real quantity \( k \) is the wavenumber and \( \omega \) is the time constant, in general, a complex number \( (\omega = \omega_R + i\omega_I) \). If \( \omega_R > 0 \), the disturbance decays while for \( \omega_R < 0 \) it grows. The marginal case is characterized by \( \omega_R = 0 \). If \( \omega_I = 0 \), the marginal case is non-oscillatory and if \( \omega_I \neq 0 \), oscillatory. The linear system for the amplitudes is:

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \Delta^2 + \frac{\omega}{Pr}\Delta \right) \Psi + \frac{i k}{Pr} \hat{H} \Psi + (i k Ra - Ra_m \widehat{Q}) \Theta - Ra_m \hat{P} \Phi = 0, \\
(\Delta + \omega) \Theta - ikU \Theta + (D - ik DT_0) \Psi = 0, \\
(D^2 - k^2) \Phi - \varepsilon H (\sin \phi D + ik \cos \phi) \Theta = 0,
\end{array} \right.
\end{align*}
\]

(11)

where the following operators are introduced

\[
\begin{align*}
D &= \frac{d}{dz}, \\
\Delta &= D^2 - k^2, \\
\hat{H} &= D^2 u_0(z) - u_0(z) \Delta, \\
\hat{P} &= k^2 DT_0(z), \\
\widehat{Q} &= ik[\sin 2\phi + \sin^2 \phi DT_0(z)] - \sin^2 \phi D.
\end{align*}
\]

(12)

The system (11) satisfy the boundary conditions

\[
\begin{align*}
\Psi(-1) = \Psi(1) = D\Psi(-1) = D\Psi(1) = 0, \\
\Theta(-1) = \Theta(1) = 0,
\end{align*}
\]

(13)

\[
\begin{align*}
D\Phi(-1) - k\Phi(-1) = D\Phi(1) + k\Phi(1) = 0.
\end{align*}
\]

The last boundary conditions (13) are derived precisely in [7] and differ from the corresponding conditions in [6] by exchanging the sign \((-\) and \((+)\), taken from [3], but it has no distinguishable effect on the final results. The problem
(11), (13) was solved by the Galerkin method ([2], chapter 1). The calculations were based on trial functions which are eigenfunctions of the stability problem for a heated quiescent liquid layer subjected to a magnetic field. The number of trial functions was varied between 10 and 20. At given $Ra_m$, we seek the minimum positive $Ra$ for which the system has a non-zero solution at $\omega^R = 0$.

Some numerical results for a flow of ferrofluid EMG 901 in a channel of half width of $d = 1$ mm, with $\beta = 1$ Kmm$^{-1}$, $H^c = 4.8 \times 10^6$ Am$^{-1}$, $K = 29$ AK$^{-1}$ and Pr = 79.3 [1] are given in Figs. 1 and 2. In Fig. 1, the decrement $\omega^R$ is plotted as a function of $Ra$ for $Ra_m = 1.28$, $k = 2.3$ and $\phi = 3\pi/4$ or $\phi = -\pi/4$. At $Ra = 0$, $\omega^R \equiv \nu_i$ are the eigenvalues of the thermal problem ([2], p. 18). For small $Ra$, the relative phase rates $\omega^R/k$ are very small which corresponds to monotonic damping of the temperature perturbations. When increasing $Ra$, the decrements $\omega^R$ merge into complex conjugated pairs (curves a,b,c and d). These pairs determine two waves of equal phase rates going in opposite directions along the channel. The curve $\nu_1$ crosses the abscissa at $Ra = 89.86$. For the Rayleigh numbers higher than this value oscillatory modes can appear.

In Fig. 2 marginal stability curves $Ra(k)$ are presented for the same $Ra_m$ and different values of the inclination angle. For any non-zero inclination of the magnetic field, the critical wavenumber is equal to 2.3 and the critical Rayleigh number varies slightly. Its smallest value is about 90 calculated for $\phi = 3\pi/4$ or $\phi = -\pi/4$. The critical $Ra$ is not influenced by the change of the direction of the magnetic field for any fixed inclination. This tendency remains
for other magnetic Rayleigh numbers less than 1.28.

For horizontal magnetic field, the critical wavenumber \( k \) is about 4 coinciding with the value for Newtonian viscous liquid. Then, the critical Ra is equal to 487, which is almost half of the corresponding number \( \text{Ra}_c = 964 \) found in [2]. The last value is obtained by us at \( \text{Pr} = 10 \) only. Although the basic flow does not depend on the magnetic field when it is parallel as well as vertical, the flow instability does. Whatever the inclination angle, the magnetic field has a quite large instability effect on ferrofluids, in comparison with the instability for usual Newtonian liquids.

4. Conclusion

The linear stability of parallel flows of a ferrofluid in a horizontal channel, subjected to a longitudinal temperature gradient and an oblique strong magnetic field, is studied by the Galerkin method. It is shown that the magnetic field has a large instability effect on ferrofluids, in comparison with Newtonian liquids. The instability appears to be oscillatory for large Prandtl numbers typical for the magnetic liquids.

REFERENCES


