ON MULTILEVEL ITERATIVE METHODS FOR NAVIER-STOKES PROBLEMS

P. BOYANOVA
Institute for Parallel Processing, Bulgarian Academy of Sciences,
25A, Acad. G. Bonchev, BG-1113 Sofia, Bulgaria,
e-mail:petia@parallel.bas.bg

S. MARGENOV
Institute for Parallel Processing, Bulgarian Academy of Sciences,
25A, Acad. G. Bonchev, BG-1113 Sofia, Bulgaria,
e-mail:margenov@parallel.bas.bg

Abstract. Two projection schemes which are based on nonconforming
Crouzeix-Raviart finite element approximation of the velocities and
piece-wise constant approximation of the pressure are considered. The im-
portant advantage of these approximations is that the divergence of the
velocity field is zero inside each element, i.e. the approximation is locally
conservative. We show that optimal order Algebraic MultiLevel Itera-
tion (AMLI) preconditioners can be successfully applied to the decoupled
scalar elliptic problems at the Prediction step as well as to the mixed
FEM problem at the Projection step. As a result, the related composite
time-stepping solution methods have a total computational complexity of
optimal order.

Key words: time dependent Navier-Stokes, nonconforming FEM,
projection scheme, multilevel PCG solver.

1. Introduction

Let us consider the Dirichlet initial-boundary value problem for the
Navier-Stokes equations

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \quad (x, t) \in \Omega \times (0, T)
\]

\[
\nabla \cdot \mathbf{u} = 0 \quad (x, t) \in \Omega \times (0, T)
\]

\[
\mathbf{u} = 0 \quad (x, t) \in \Gamma \times (0, T)
\]

\[
\mathbf{u} = 0 \quad (x, t) \in \Omega \times \{0\}
\]

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where $\Omega$ is a bounded and connected domain in $\mathbb{R}^2$, and $\Gamma = \partial \Omega$. We assume also that $\Omega$ is such that the $H^2$-regularity property [3] holds for the steady Stokes problem. As is well known, a generalization to nonhomogeneous boundary condition is straightforward.

The numerical solution of the incompressible Navier-Stokes equations has been the focus of the computational fluid dynamics community for over five decades. However, the question how to construct optimal schemes, in terms of computational cost and accuracy, is still not over. What is quite clear now is that the solution via Uzawa iterations of the coupled velocity-pressure discrete systems that result from the space and time discretization of the equations is quite expensive. The most popular alternative way to build efficient approximations is known as projection approach. It is based on a stable splitting of each time step.

Let us assume that $\Omega$ is a polygonal domain, and $\mathcal{T}_h$ is a triangulation of $\Omega$. In this section we will use also the following notations: $\mathcal{V}^c$ and $\mathcal{V}^{nc}$ are respectively the linear conforming (Courant) and linear nonconforming (Crouzeix-Raviart) FE spaces satisfying homogeneous boundary conditions; $\mathcal{Q}$ is the space of piece-wise constant pressures, i.e.,

$$\mathcal{Q} = \left\{ q \in L^2(\Omega) : q|_e \in P_0, \forall e \in \mathcal{T}_h, \int_{\Omega} q = 0 \right\},$$

($...,;...$) and ($...,e$) stand for the dot products in $L^2(\Omega)$ and $L^2(T)$; and ($...,h$) = $\sum_{e \in \mathcal{T}_h} (...,e)$. Uniform discretization in time is used with a time step $\Delta t$. The upper index in the presented projection schemes stands for the number of the time discretization level. For example, $(u^n_h, p^n_h)$ are the numerically computed velocities and pressure for $t = n\Delta t$.

Now we will consider two projection schemes (Variant A and Variant B) which are based on Crouzeix-Raviart FE approximation of the velocities and piece-wise constant approximation of the pressure. The most significant advantage of these approximations is that the divergence of the velocity field is zero inside each element, i.e. the approximation is locally conservative. Note that this is not the case, e.g., for the alternatively applied, inf-sub stable approximation for the velocities, using rotated bilinear (Rannacher-Turek) elements.

**Variant A:**

This scheme is based on a complete nonconforming discretization of the velocities, and is inf-sub stable and locally conservative [9].

- **Prediction step:**
Find $\mathbf{u}^{n+1}_h \in (\mathcal{V}^{nc})^2$ such that for all $\mathbf{v}_h \in (\mathcal{V}^{nc})^2$

$$
\left( \frac{\mathbf{u}^{n+1}_h - \mathbf{u}^n_h}{\Delta t}, \mathbf{v}_h \right) + \left( (\mathbf{u}^n_h, \nabla) \mathbf{u}^n_h, \mathbf{v}_h \right)_h + \frac{1}{Re} (\nabla \mathbf{u}^{n+1}_h, \nabla \mathbf{v}_h)_h - (p^n_h, \nabla \cdot \mathbf{v}_h)_h = 0
$$

(2)

- **Projection step:**
  Find $\mathbf{u}^{n+1}_h \in (\mathcal{V}^{nc})^2$, $p^{n+1}_h \in \mathcal{Q}$ such that

$$
\left( \mathbf{u}^{n+1}_h - \mathbf{u}^n_h, \mathbf{v}_h \right) = (p^{n+1}_h - p^n_h, \nabla \cdot \mathbf{v}_h)_h, \quad \forall \mathbf{v}_h \in (\mathcal{V}^{nc})^2
$$

(3)

$$
(\nabla \cdot \mathbf{u}^{n+1}_h, q_h)_h = 0, \quad \forall q_h \in \mathcal{Q}
$$

**Variant B:**

Conforming FEs at the Prediction step are used to reduce the computational complexity. The accuracy of the velocities is of optimal order subject to the "cross-grid mesh" condition [3].

- **Prediction step:**
  Find $\mathbf{u}^{n+1}_h \in (\mathcal{V}^c)^2$ such that for all $\mathbf{v}_h \in (\mathcal{V}^c)^2$

$$
\left( \frac{\mathbf{u}^{n+1}_h - \mathbf{u}^n_h}{\Delta t}, \mathbf{v}_h \right) + \left( (\mathbf{u}^n_h, \nabla) \mathbf{u}^n_h, \mathbf{v}_h \right)_h + \frac{1}{Re} (\nabla \mathbf{u}^{n+1}_h, \nabla \mathbf{v}_h)_h - (p^n_h, \nabla \cdot \mathbf{v}_h)_h = 0
$$

(4)

- **Projection step:**
  Find $\mathbf{u}^{n+1}_h \in (\mathcal{V}^{nc})^2$, $p^{n+1}_h \in \mathcal{Q}$ such that

$$
\left( \mathbf{u}^{n+1}_h - \mathbf{u}^n_h, \mathbf{v}_h \right) = (p^{n+1}_h - p^n_h, \nabla \cdot \mathbf{v}_h)_h, \quad \forall \mathbf{v}_h \in (\mathcal{V}^{nc})^2
$$

(5)

$$
(\nabla \cdot \mathbf{u}^{n+1}_h, q_h)_h = 0, \quad \forall q_h \in \mathcal{Q}
$$

The advantage of Variant A is the inf-sub stability while Variant B is computationally cheaper due to the three times reduction of the unknowns at the Prediction step. In both variants, the projection scheme splits the nonlinear Navier-Stokes system to some of the following linear elliptic problems.

- **Variant A - Prediction step:**
  For a given $f \in L^2(\Omega)$ find $\mathbf{u}_h \in \mathcal{V}^{nc}$, satisfying

$$
\frac{1}{\Delta t} (\mathbf{u}_h, \mathbf{v}_h) + \frac{1}{Re} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_h = (f, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathcal{V}^{nc}.
$$

(6)
• **Variant B - Prediction step:**
  For a given \( f \in L^2(\Omega) \) find \( u_h \in V^c \), satisfying
  \[
  \frac{1}{\Delta t} (u_h, v_h) + \frac{1}{Re} (\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V^c.
  \]

• **Variants A,B - Projection step:**
  For a given \( f \in (L^2(\Omega))^2 \) find \((u_h, p_h) \in (V^{nc}, Q)\), satisfying
  \[
  (u_h, v_h) - (p_h, \nabla \cdot v_h)_h = (f, v_h) \quad \forall v_h \in V^{nc},
  
  (\nabla \cdot u_h, q_h)_h = 0 \quad \forall q_h \in Q.
  \]

Optimal order robust AMLI preconditioners for problems (6) and (7) can be constructed as generalizations of the results presented, e.g., in [5] and [1]. One could expect that the AMLI methods are not directly applicable to the saddle point system which arises from the mixed FEM problem (8). A solution of this problem was recently proposed in [6].

The next sections are devoted to the theory and some numerical tests for a class of optimal AMLI algorithms for the linear systems mostly focused on Variant A.

2. Robust AMLI preconditioners

2.1. The AMLI method

AMLI is a recursive generalization of two-level preconditioners that has optimal computational complexity due to a proper Schur complement stabilization using Chebyshev polynomials, see, e.g., [2, 10]. Let us consider a sequence of nested triangulations \( T_0 \subset T_1 \subset \cdots \subset T_l \) of the domain \( \Omega \), constructed by recursive uniform refinement of a given initial mesh. We denote by \( A(0), A(1), \ldots, A(l) = A \) the corresponding system matrices. Regarding a hierarchical 2×2 block partitioning of the matrix \( A(k) \) at a refinement level \( k \),

\[
\widetilde{A}(k) = J(k)^T A(k) J(k) \quad \text{DOF added by refinement}
\]

\[
\begin{bmatrix}
\widetilde{A}_{11}(k) & \widetilde{A}_{12}(k) \\
\widetilde{A}_{21}(k) & \widetilde{A}_{22}(k)
\end{bmatrix} \quad \text{coarse mesh DOF}
\]

\((J(k))\) is the hierarchical basis transformation and DOF stands for the related degrees of freedom) the AMLI preconditioner is defined as follows: \( C(0) = A(0) \) at the coarsest mesh level with index 0,

\[
C(k) = J(k)^{-1} \begin{bmatrix}
\widetilde{C}_{11}(k) & 0 \\
A_{21}^{-1} & \widetilde{A}(k-1)
\end{bmatrix}
\begin{bmatrix}
I & \widetilde{C}_{11}^{-1} \widetilde{A}_{12}(k) \\
0 & I
\end{bmatrix} J(k)^{-T}
\]
at successively refined levels $k$, where $\tilde{C}_{11}^{(k)}$ are symmetric positive definite approximations of $\tilde{A}_{11}^{(k)}$ that satisfy

\begin{equation}
{v^T A_{11}^{(k)} v \leq v^T C_{11}^{(k)} v \leq (1 + b)v^T \tilde{A}_{11}^{(k)} v, \text{ for all } v,}
\end{equation}

and $\tilde{A}^{(k-1)^{-1}} = Q_{\beta - 1} \left( C^{(k-1)^{-1}} A^{(k-1)} \right) C^{(k-1)^{-1}}$, where $Q_{\beta - 1}$ is a properly chosen polynomial of degree $\beta - 1$. It is known that under conditions, detailed in [2], the relative condition number satisfies the relation

\begin{equation}
\kappa(C^{(l)^{-1}} A^{(l)}) \approx \frac{1 + b}{1 - \gamma^2},
\end{equation}

where $\gamma$ is the Cauchy-Bunyakowski-Schwarz (CBS) constant, that characterizes the two-level hierarchical partitioning. For the case $\beta = 2$ the coefficients $q_0$ and $q_1$ of the optimal stabilization polynomial $Q_1(y) = q_0 + q_1 y$, which has to be evaluated in the described AMLI algorithm, are given by

\begin{equation}
q_0 = \frac{2}{\xi}, \quad q_1 = \frac{-1}{1 - \gamma^2 + b(1 - 2\xi)}, \quad \text{with} \quad \xi = \sqrt{1 + b + b^2 - \gamma^2 - b}.
\end{equation}

Let us note that the CBS constant $\gamma$ has a key role for both, the convergence estimate (10) and the construction of AMLI parameters (11). The general theory says that if

\begin{equation}
\frac{1}{\sqrt{1 - \gamma^2}} < \beta < \varrho
\end{equation}

then the AMLI algorithm has optimal computational complexity, i.e. proportional to the number of DOF at the finest level $\ell$. Here $\varrho$ stands for the mesh refinement ratio. In the considered 2D case $\varrho \approx 4$.

### 2.2. AMLI preconditioning of Prediction-Projection systems

**Prediction step:**

The implementation of Prediction step involves two decoupled problems with the following system matrix $A = A^{(\ell)}$

\begin{equation}
A = \frac{1}{\Delta t} M + \frac{1}{Re} K.
\end{equation}

Here $M = M^{(\ell)}$ and $K = K^{(\ell)}$ stand for the mass and stiffness matrices of the nonconforming Crouzeix-Raviart finite elements. It is important to note that in the case of nonconforming elements, the finite element spaces related to two nested triangulations are not nested. Then, some proper (locally defined) basis transformation is needed to get a hierarchical $2 \times 2$ splitting of the matrix. Here we assume that the DA splitting introduced in [4] is used.
Lemma 1 The constant in the strengthened CBS inequality corresponding to the DA splitting of the system matrix (13) is bounded by

\[ \gamma^2 < \frac{3}{4}. \]

The estimate is uniform with respect to both, the refinement level \( k \) (or the mesh parameter \( h \)) and the mesh anisotropy of the initial triangulation \( T_0 \).

Lemma 1 is a generalization of the robust CBS constant estimate from [4] where a scalar elliptic problem is considered. The proof follows straightforwardly from the related result for parabolic problems, see [8]. It is based on the inequality \( \gamma = \gamma_A \leq \max\{\gamma_M, \gamma_K\} \) where \( \gamma_M^2 < 1/2, \gamma_K^2 < 3/4 \).

Applying Lemma 1 we get that the optimality conditions (12) for the system matrix (13) are satisfied if the acceleration polynomial has a degree \( \beta \in \{2, 3\} \). This means that the Prediction step can be implemented efficiently solving iteratively the related linear systems by PCG method using the AMLI preconditioner. The number of AMLI iterations is uniformly bounded - only few iterations are usually needed independently of the number of DOF. At the same time, the computational complexity of each AMLI iteration is optimal, i.e. proportional to the number of DOF.

Projection step:
As we already mentioned, the AMLI method is not directly applicable to the Projection step. The problem is that the related system matrix has the saddle point form

\[ P = \begin{bmatrix} M & B_1 \\ B_1^T & M \end{bmatrix}. \]

Here, \( M = M^{(\ell)} \) is the mass matrix corresponding to the scalar Crouzeix-Raviart FEM space which is used to approximate the velocity components. It is very important to note, that \( M \) is diagonal, since the quadrature formula on a triangle with nodes in the midpoints of the edges is exact for second degree polynomials. This allows us to eliminate exactly (locally) the velocity unknowns reducing the problem to a system for the pressure unknowns only. The related Schur complement matrix \( S = S^{(\ell)} \) has the form

\[ S = B_1^T M^{-1} B_1 + B_2^T M^{-1} B_2. \]

Due to specific sparsity structure, it is referred to as weighted graph-Laplacian. The construction of AMLI preconditioner for \( S \) meets new challenges. A new
concept of element matrices of graph-Laplacians was introduced in [11] in order to construct and analyze proper hierarchical two-level splittings. More recently, a parametric family of AMLI algorithms for weighted graph-Laplacians (16) was studied in [6].

**Lemma 2** Let us consider the parametric multilevel splitting of the weighted graph-Laplacian (16) defined in [6] in the case of mesh \( T_0 \) consisting of isosceles rectangle triangles. Then, the CBS constant corresponding to parameters \( p=1, q=-0.1, t=0.75 \) is bounded by

\[
\gamma^2 \leq 0.58,
\]

uniformly with respect to the refinement level \( k \).

The proof presented in [6] is based on edge-related local analysis. Similarly to the Prediction step, Lemma 2 means that the AMLI algorithm with \( \beta \in \{2, 3\} \) will satisfy the optimality conditions (12).

3. Numerical tests

The presented numerical tests illustrate the optimal convergence rate of the AMLI algorithm for iterative solution of systems with weighted graph-Laplacian (15). A model problem in \( \Omega = (0,1)^2 \) is considered. The number of DOF at \( T_0 \) is \( N_0 = 2^{2(4+\ell)+1} \), starting with a \( 2 \times 16 \times 16 \) mesh \( T_0 \). According to the theory, the number of iterations is independent of the DOF for a given relative error \( \epsilon \) of the PCG stopping criteria.

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Table 1 is for the case where the best polynomial approximation of \( [\hat{A}_{11}^{(k)}]^{-1} \) is used [7]. The results fully confirm the theoretical estimates. Table 2 illustrates one alternative variant where the approximate action of \( [\hat{A}_{11}^{(k)}]^{-1} \) is given by few CG iterations. The advantages of the first approach are well expressed.

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4. Concluding remarks

Two inf-sub stable and locally conservative (divergence-free) projection schemes are considered. The developed AMLI preconditioners for both, Prediction and Projection steps, provide composite algorithms (Navier-Stokes solvers) of optimal computational complexity.

REFERENCES


